A Graphical Analysis of the Cournot-Nash and Stackelberg Models

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Graphical analysis plays an important role in economic analysis, allowing a comparison to be easily made among different types of problems and providing a method of developing the intuition underlying economic concepts. For instance, the standard supply and demand framework allows a direct comparison of price, quantity, and economic welfare in the polar cases of competition and monopoly. Drawing the demand curve and marginal revenue curve together also provides the intuition behind the results of monopoly and competition.

The polar cases of monopoly and competition, however, do not represent the complete set of possible market outcomes. In the industrial organization literature, for instance, the Cournot-Nash and Stackelberg solutions are widely used. To date, however, the price, quantity, and economic welfare associated with these models are not typically compared with those of monopoly and perfect competition in the standard price-quantity framework. Instead, the analysis of the industrial organization models has traditionally focused on reaction curves in a quantity-quantity framework. This shift in the framework of analysis makes it difficult for undergraduate students to see the impact of these alternative equilibrium notions on price, quantity, and economic welfare.

There are some exceptions to this standard treatment of industrial organization models. Gravelle and Rees (1981; 313) present an analysis in which the linkage is made between the standard price-quantity framework and the quantity-quantity framework. However, they do not specifically derive the reaction functions, nor do they show how the Cournot-Nash solution compares with the competitive and monopolistic solutions. Although Martin (1988; chapt. 5) does provide an analysis of this sort (in fact, the analysis of the two-firm Cournot-Nash model in this article parallels his presentation), neither he nor Gravelle and Rees expand the Cournot-Nash model beyond two firms or develop the Stackelberg model. In this article, I outline how the Cournot-Nash and Stackelberg equilibria can be represented in the familiar supply-demand framework. One advantage of these described methods is that they allow a direct comparison among the monopoly, competitive, and industrial organization models. Another advantage is that the economic intuition underlying these models can be much better understood. Finally, because the methods can handle relatively complicated models in which firms have different cost curves, a wide range of problems is opened to analysis.

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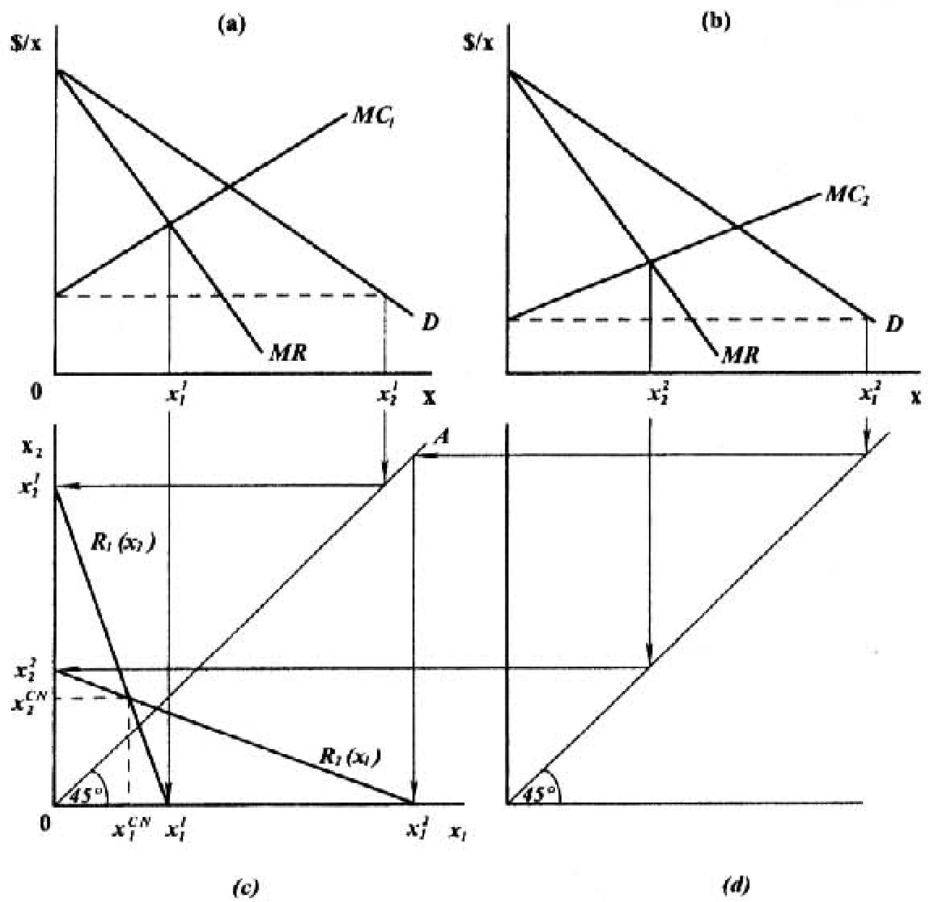
In the next section, I examine the Cournot-Nash model, first with an analysis that considers two firms with different marginal cost curves and second for the case of N firms with identical marginal costs. The Stackelberg model is then presented with an examination of a two-firm model and an (N + 1)-firm model.

COURNOT-NASH MODEL[1]

Consider an industry with two firms. Panel (a) in Figure 1 shows the marginal cost curve (MC_1) for firm 1, and panel (b) shows the marginal cost curve (MC_2) for firm 2. The marginal costs of the two firms need not be identical. The curve D in both panels is the industry demand curve facing the two firms. The marginal cost curves and the demand curve are assumed to be linear.

In the Cournot-Nash model, the assumption is that each firm believes the other firm will not change its output. As a consequence, each firm is able to determine the profit-maximizing level of output to produce, given the output of the other firm. The relationship between the profit-maximizing level of output of one firm and the given level of output by the other firm is called the reaction function. When the marginal cost and demand curve are linear as in Figure 1, the reaction function is also linear. As a result, only two combinations of the profit-maximiz-

FIGURE 1
Derivation of Reaction Functions from Marginal Cost and Demand Curves



ing level of output of one firm and the given output of the other firm need to be found to specify the entire reaction function.

To find the profit-maximizing output level of one firm, given a fixed level of output by the other firm, one should consider the notion of a *ceteris paribus* demand curve (Gravelle and Rees 1981; 311). The *ceteris paribus* demand curve of firm 1 is the market demand curve D shifted to the left by the amount of output firm 2 is producing. Similarly, the *ceteris paribus* demand curve of firm 2 is the market demand curve D shifted to the left by the output produced by firm 1.

Consider firm 1 first. If firm 2 is producing zero output, firm 1 faces the entire industry demand curve. The profit-maximizing level of output for firm 1 to produce in this case is x_1^1, which is determined by the intersection of MR and MC_1 (Figure 1, panel a). This means the combination (x_1^1, 0) is a point on firm 1's reaction function $R_1(x_2)$ (see panel c).

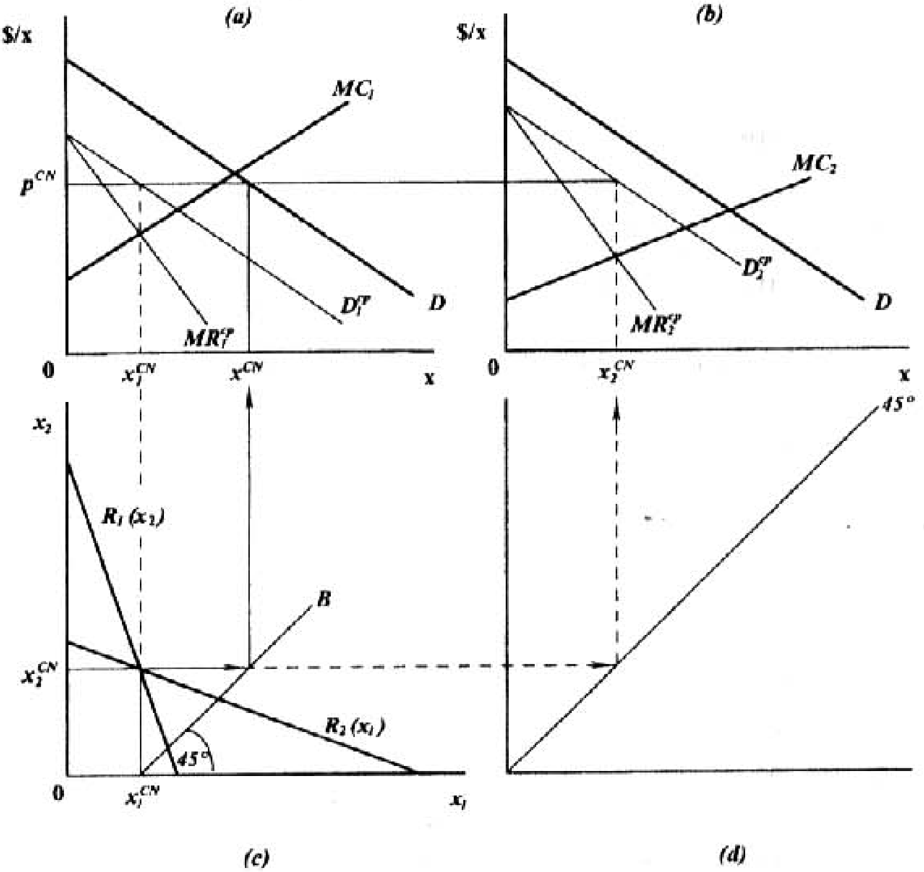
A second point on firm 1's reaction function is found by considering the level of output firm 2 would have to produce to cause firm 1 to produce zero output. If firm 2 produces output level x_2^1, then the profit-maximizing level of output for firm 1 is zero (panel Figure 1, a). Output x_2^1 is the level of output that shifts firm 1's *ceteris paribus* demand curve down to the point where the demand curve no longer cuts the marginal cost MC_1. Thus, the combination (0, x_2^1) is also a point on the reaction function of firm 1 (panel c). Mapping x_2^1 from panel (a) to panel (c) can be done using the 45° line OA in panel (c).[2]Linking point (0, x_2^1) with point (x_1^1, 0) gives the reaction function R_1x_2.

The reaction function for firm 2 can be found in a similar fashion. If firm 1 is producing zero output, firm 2 faces the entire industry demand curve. The profit-maximizing level of output for firm 2 is x_2^2, which is determined by equating MR with MC_2; thus, point (0, x_2^2) is a point on firm 2's reaction function R_2x1. If firm 1 is producing output level x_1^2, then firm 2 will produce zero output. Thus, (x_1^2, 0) is also a point on firm 2's reaction function.[3] Connecting the points (0, x_2^2) and (x_1^2, 0) gives the reaction function R_2x_1. The intersection of R_1x_2 and R_2x_1 gives the Cournot-Nash equilibrium values, x_1^{CN} and x_2^{CN}. Because firm 2 has the lower marginal cost curve, its output in equilibrium is greater than that of firm 1.

The total amount of output and the price associated with the Cournot-Nash equilibrium is shown in Figure 2. Total output, x^{CN}, is the sum of x_1^{CN} and x_2^{CN} (panel a). The price associated with output x^{CN} is given by p^{CN}. Figure 2, panel (c), shows how total output is determined graphically. The vertical axis in panel (c) indicates firm 2's production (x_2), while the horizontal axis shows firm 1's production (x_1). To find the sum of x_1 and x_2, consider a 45° line running northeast of any point (x_1, 0). This line maps quantity x_1 onto quantity $x_1 + x_2$. For instance, consider the point (x_1^{CN}, 0). At this point, the 45° line $x_1^{CN}B$ cuts the horizontal axis and the sum $x_1^{CN} + x_2^{CN}$ equals x_1^{CN}. In a similar fashion, quantity x_2^{CN} can be mapped onto quantity $x_1^{CN} + x_2^{CN}$; graphically, a horizontal line at (0, x_2^{CN}) intersects the 45° line $x_1^{CN}B$ at a horizontal distance of $x^{CN} = x_1^{CN} + x_2^{CN}$.

It is useful to show that x_1^{CN} and x_2^{CN} are in fact the Cournot-Nash equilibrium output levels for firm 1 and firm 2. A Cournot-Nash equilibrium is defined as the set of outputs for two firms such that when one firm produces its equilibrium level of output, the profit-maximizing level of output for the other firm is its equi-

FIGURE 2
Prices and Quantities in a Cournot-Nash Equilibrium



librium level of output. For example, suppose x_1* is the profit-maximizing level of output for firm 1 given that firm 2 is producing x_2* and that, simultaneously, x_2* is the profit-maximizing level of output for firm 2 given that firm 1 is producing x_1*. If this is the case, then (x_1*, x_2*) is a Cournot-Nash equilibrium, since if firm 1 and firm 2 actually produce x_1* and x_2*, respectively, there is no incentive for either firm to change its level of production.

This equilibrium concept is illustrated in Figure 2. With firm 2 producing x_2^{CN}, firm 1 finds that its *ceteris paribus* demand curve becomes D_1^{cp} (curve D_1^{cp} is derived by shifting curve D to the left by an amount x_2^{CN}). The marginal revenue curve to this demand curve is MR_1^{cp}. Equating MR_1^{cp} with MC_1 gives x_1^{CN} as the profit-maximizing level of output. Similarly, with firm 1 producing x_1^{CN}, firm 2's *ceteris paribus* demand curve becomes D_2^{cp} (curve D_2^{cp} is derived by shifting curve D to the left by an amount x_1^{CN}). The marginal revenue curve to this demand curve is MR_2^{cp}. Equating MR_2^{cp} with MC_2 gives x_2^{CN} as the profit-maximizing level of output. Thus, the combination (x_1^{CN}, x_2^{CN}) is a Cournot-Nash equilibrium.

The price and quantity combination derived from the Cournot-Nash model lies between that derived from a perfectly competitive model (x^C, p^C) and that derived from a monopoly model (x^M, p^M) (Figure 3). Figure 3 also shows the price and quantity combination that results from a Stackelberg model. This model is derived in the next section. It is important to stress that while the equilibrium price and quantity in the Cournot-Nash model are on the demand curve, they are not on the supply curve, where the supply curve is defined as the horizontal sum of the marginal cost curves. Thus, just as in a monopoly, a supply curve does not exist for an oligopolistic industry.

Cournot-Nash Model with *N* Identical Firms

The above analysis considered two firms with potentially different marginal cost curves. If it is assumed the firms have identical marginal costs, it is possible to find the level of output that would be produced if more than two firms are present in the industry. To see this, consider an industry with *N* firms. Because all firms are identical, it is useful to think of one of the firms as a representative firm and to define a representative reaction curve for that firm.[4] The representative firm's reaction curve shows the profit-maximizing output for the representative firm, given the output of *all* the other firms. Let firm 1 be the representative firm, with output denoted as x_1.

Figure 4 shows the derivation of the representative firm's reaction curve. To find the representative reaction curve for firm 1, suppose all the other $(N - 1)$ firms produce nothing. In this case, firm 1 faces the entire industry demand curve. The profit-maximizing output for firm 1, $x_1{}^1$, is found by equating the marginal revenue curve *MR* with the common and representative marginal cost curve

FIGURE 3
Comparison of Competitive, Monopoly, Cournot-Nash, and Stackelberg Solutions

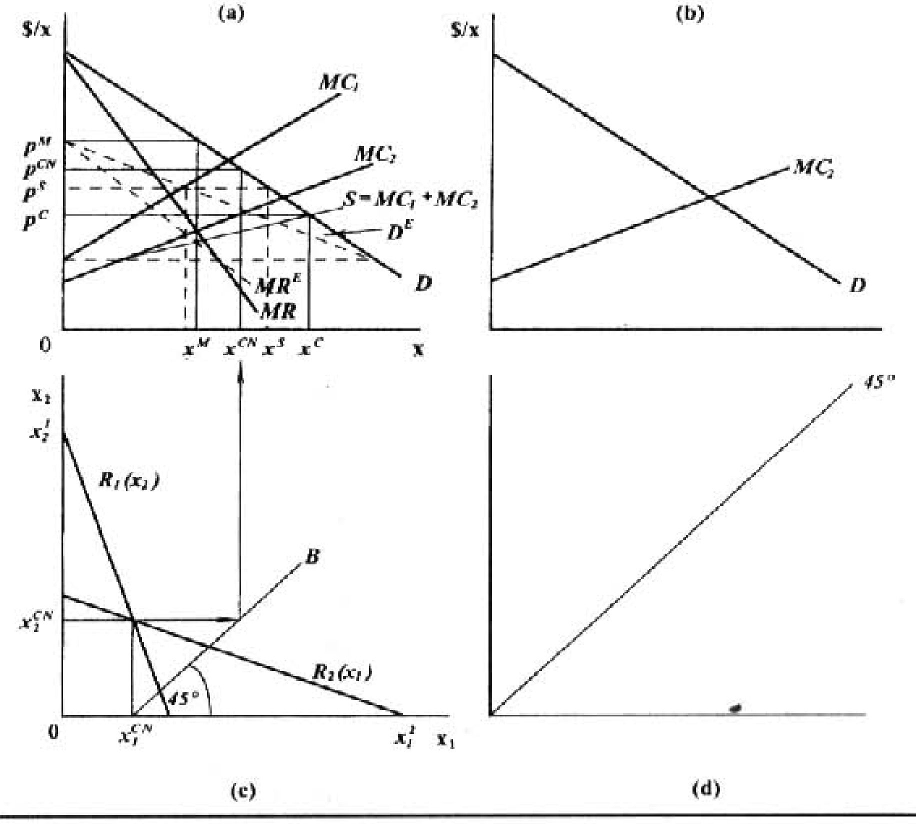
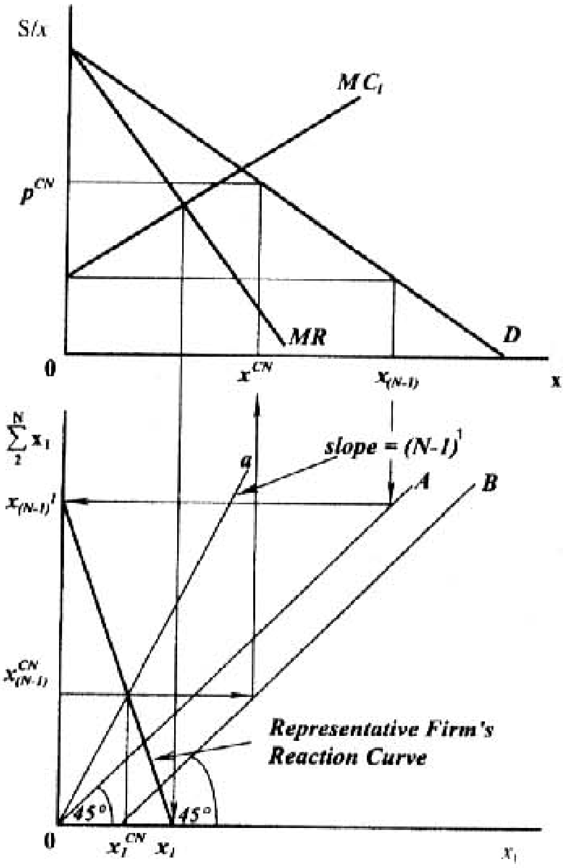


FIGURE 4
Determination of the Cournot-Nash Output in a Model with *N* Identical Firms



MC_1. The vertical axis in the bottom panel of Figure 4 shows the output of all the other firms besides the representative firm. Thus, the point x_1^1, 0 in the bottom panel is one point on the representative reaction curve. Now suppose all the other firms together produce a quantity x_{N-1}^1. Given this level of output, the profit-maximizing level of output for firm 1 is zero. Thus, the point 0, x_{N-1}^1 in the bottom panel is a second point on the representative firm's reaction curve. Linking point 0, x_{N-1}^1 with point x_1^1, 0 gives the representative firm's reaction curve.

The Cournot-Nash equilibrium can be found by noting that because all firms are identical, all will produce the same amount in equilibrium. To find the equilibrium quantity for the representative firm x_1^{CN}, locate the point where the representative firm's reaction curve cuts line 0a. Line 0a has a slope of $N - 1$. Along line 0*a*, the quantity on the vertical axis is $N - 1$ times the quantity on the horizontal axis. At the intersection of the reaction curve and line 0*a*, the output of the remaining $N - 1$ firms is $(N - 1)$ times that of the representative firm. Thus, all firms are producing the same output. The equilibrium total output of all *N* firms is $x^{CN} = Nx_1^{CN}$. Total output can be determined graphically by finding the point where a horizontal line at 0, x_{N-1}^{CN} intersects the 45° line $x_1^{CN}B$. The horizontal distance $x^{CN} = x_1^{CN} + x_{N-1}^{CN}$ is the total output under the Cournot-Nash equilibrium.

To see how this method might be used, suppose N equals three. Thus, line $0a$ in Figure 4 has a slope of 2. The intersection of line $0a$ with the representative firm's reaction curve indicates that firm 1's Cournot-Nash equilibrium output level is x_1^{CN}. The remaining $(N - 1)$ firms (two firms in this example) produce x_{N-1}^{CN}. Total industry output will be x^{CN} and the price will be p^{CN}. This price and quantity can be compared with the monopoly and competitive price and quantity.

Because the derivation of the representative firm's reaction curve does not depend on N, the impact of altering the number of firms in the industry can easily be examined by changing the slope of line $0a$. Similarly, for a given N, changes in the parameters of the demand curve or marginal cost curve can be found by rederiving the location of the representative firm's reaction curve.

Stackelberg Model

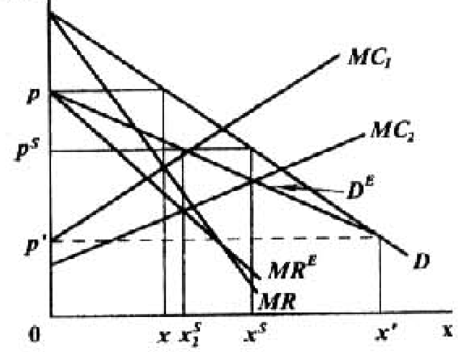
In the Stackelberg model, one of the firms is a leader while the others are followers. The leader knows how the followers will react and takes account of this in deciding how much output to produce. It is usually assumed that followers adopt Cournot-Nash behavior (i.e., each of the followers believes the leader and the other followers will not change their level of output).

To examine the solution to the Stackelberg model in which there is one leader and one follower, consider Figure 5. The demand curves and marginal cost curves are the same as in Figure 1. To aid in the analysis, I drew these curves on the same graph. Firm 2 is assumed to be the leader.

How should the leader choose its output level? The key to answering this question is to think about the *price* that will result for different levels of output by the leader. Suppose the leader produces no output. In this case, the follower (firm 1) has the entire market and will produce a level of output x (found by equating MR with MC_1). The resulting price is p. Thus, the combination $(0, p)$ represents one output-price combination the leader can potentially face .

Suppose the leader (firm 2) decides to produce a level of output equal to x'. At this level of output, the profit-maximizing level of output for the follower is zero.

FIGURE 5
Derivation of the Stackelberg Solution in a Two-Firm Industry



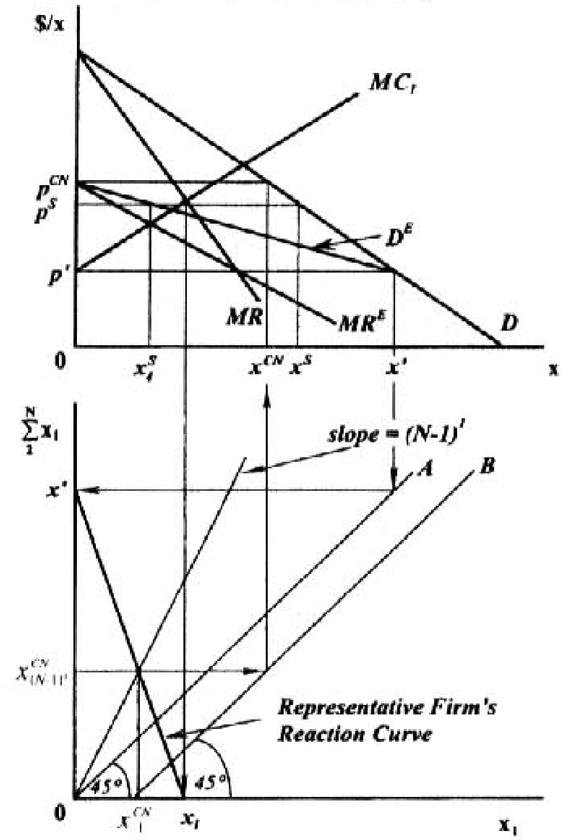
This means the price is p'. Thus, a second output-price combination the leader can potentially face is given by the point (x', p'). Drawing a line between these points gives the effective demand curve (D^E) facing the leader.[5] The marginal revenue curve to this demand curve is MR^E (Figure 5).

Equating the marginal revenue curve MR^E with marginal cost MC_2 gives the level of output (x_2^S) that maximizes profits for firm 2, the leader.[6] The price that results is p^S. Firm 1 (the follower) produces the difference between total output (x^S) and the output produced by the leader. Total output x^S is the quantity that gives rise to price p^S using the market demand curve D.[7]

The price and quantity combination in the Stackelberg case can be compared with those in the competition, monopoly, and Cournot-Nash models (Figure 3). Given linear demand and supply curves, the price generated in a Stackelberg model will be less than the price in the Cournot-Nash model, although it will be greater than the price in the competitive model. The concepts of consumers' and producers' surplus can be used to examine the distributional effects of the different types of firm behavior.

The Stackelberg model can also be extended to analyze an industry with more than two firms. Suppose there are $N + 1$ firms in an industry. In this case, N of the firms will be followers, while one will be the leader. Assume the marginal cost curves of all the followers are identical.

FIGURE 6
Derivation of the Stackelberg Solution in a (N + 1)-Firm Industry



If the leader produces nothing, the output of the N followers can be determined using the methodology outlined in Figure 4. If the resulting price is denoted as p^{CN}, then the point 0, p^{CN} is one point on the effective demand curve of the leader. Another point on the effective demand curve of the leader can be found by determining the level of output the leader must produce (denote this as x') so that there is no market left for the followers. If the price associated with x' is denoted as p', then the point x', p' represents a second point on the effective demand curve of the leader. Equating the marginal revenue curve to this effective demand curve with the marginal cost curve of the leader gives the profit-maximizing level of output for the leader.

As an example, suppose $N = 3$, that is, there are three followers and one leader. Assume that the marginal cost curves and the demand curve are the same as in Figure 4 (these are reproduced in Figure 6). If the leader (firm 4) produces nothing, then the profit-maximizing level of output of the three followers is given by the analysis in Figure 4. Thus, the point 0, p^{CN} is one point on the effective demand curve of the leader. A second point on the effective demand curve of the leader is the quantity and price combination x', p'. If the leader produces quantity x', then there is no market left for the followers; hence the market price is p'.

To maximize profits, the leader equates the marginal revenue curve MR^E with its marginal cost curve. If the marginal cost curve for the leader is the same as that for the followers, then the profit-maximizing level of output for the leader is x_4^S. The resulting price is p^S. The analysis can also accommodate the case where the marginal cost curve of the leader is different from that of the followers.

CONCLUDING REMARKS

Although the Cournot-Nash and Stackelberg equilibria are widely used in the industrial organization literature, these concepts are not widely understood by introductory economics students, partly because the traditional treatment of these concepts is carried out in the unfamiliar quantity-quantity framework.

I present a method for depicting the Cournot-Nash and Stackelberg equilibria in the standard supply and demand framework. The benefits of this approach are that students can easily compare the efficiency and the distributional impacts of these different market structures, and students may better understand the intuition behind the industrial organization models. In addition, the availability of a simple graphical tool that handles industrial organization concepts means a large number of additional topics can be opened up to students, such as models of industry entry (the Stackelberg model) and models of strategic trade (the Cournot-Nash model and the Stackelberg model).[8]

NOTES

1. The methodology developed in this section parallels that of Martin (1988, chap. 5).
2. Because points along a 45° line represent equal values on both the vertical and horizontal axes, the 45° line can be used to map any quantity on the horizontal axis to an equivalent quantity on the vertical axis (and vice-versa). Point $(x_2^1, 0)$ in panel (a) of Figure 1 can thus be mapped onto point $(0, x_2^1)$ in panel (c) using the 45° line in panel (c). The arrows in panel (c) show how this mapping is done.

3. Point $(x_2^2, 0)$ in panel (b) of Figure 1 can be mapped onto point $(0, x_2^2)$ in panel (c) using the 45° line in panel (d). Similarly, point $(x_1^2, 0)$ in panel (b) can be mapped onto point $(x_1^2, 0)$ in panel (c) using the 45° lines in panel (d) and (c). The arrows in panels (c) and (d) show how this mapping is done.
4. I would like to thank Richard Cornes for suggesting the use of the representative firm.
5. The proof of this derivation is as follows. Consider a linear inverse demand curve $p = a - b(x_1 + x_2)$, where x_1 and x_2 are the outputs of firm 1 (the follower) and firm 2 (the leader), respectively. One method of determining the Stackleberg solution is to substitute the follower's reaction function, R_1x_2, into the demand curve, that is, $p = a - b(R_1x_2 + x_2)$. The result is the market price can be expressed solely as a function of the leader's output. The curve $p = a - b(R_1x_2+x_2)$ is the effective demand curve (D^E) derived in Figure 5. When the leader produces zero output, the follower is able to act as if it is a monopolist; thus, the follower chooses output x in Figure 5. The output-price combination $(0, p)$ is thus a point on D^E. Another point on D^E can be found by determining the level of output firm 2 would have to produce in order to get firm 1 to produce zero output. This output level is x'. Assuming the marginal cost curve for firm 1 is $MC_1 = \alpha_1 + _{b1} \chi_1$, the reaction curve for firm 1 under Cournot-Nash behavior is $x_1 = a - \alpha_1 - bx_2/2b + \beta_1$. Firm 1 will produce zero output if $x_2 = a - \alpha_1/b$. Thus, $x' = a - \alpha_1/b$. and $p' = \alpha_1$. Therefore the output-price combination (x', p') is also a point on D^E. Because firm 1's reaction curve is linear and the inverse demand curve is linear, the effective demand curve D^E is also linear over the output range zero to x'. In this output range, curve D^E can be derived by joining the output-price combinations $(0, p)$ and (x', p'). For output greater than x', the effective demand curve is the market demand curve D (see Carlton and Perloff 1994; chap. 5, for a discussion of this latter point in the context of the dominant firm, competitive fringe model).
6. If the marginal cost curve of the leader is sufficiently low, the leader may set price below p', thus making the leader the sole producer in the market. See Carlton and Perloff (1994; chap. 5) for a discussion in the context of the dominant firm, competitive fringe model.
7. The Stackelberg model shares many similarities with the dominant firm, competitive fringe model. When the followers act as price takers rather than adopting Cournot-Nash behavior, the Stackelberg model reduces to the dominant firm model. A comparison can be made between the Stackelberg model and the dominant firm model using Figure 5. Suppose that instead of adopting Cournot-Nash behaviour, the follower determines output by equating price with marginal cost. If the leader produces zero output, the follower produces quantity x^c (not shown), where x^c is the quantity at which MC_1 equals D. This output is greater than output x produced when the follower adopts Cournot-Nash behavior. The output produced is greater because the follower is not using its market power. If the price associated with output x^c is denoted p^c (not shown), then combination $(0, p^c)$ represents one potential output-price combination the leader can face. The leader also faces the output-price combination (x', p'), because if the leader produces output x', the follower will produce zero output. The effective demand curve facing the leader is thus derived by linking the points $(0, p^c)$ and (x', p'). This curve is similar to curve D^E in Figure 5, except that it is rotated downwards around the point (x', p'). The marginal revenue curve is also rotated downwards in a similar fashion (the pivot point is the intersection of MR and MR^E). As a consequence, this new marginal revenue curve cuts MC_2 at an output level that is less that x_2^S, implying that the leader produces less output. Total industry output, however, is higher because the followers produce a greater quantity. The result is a lower price than under the Stackelberg model.
8. See Tirole (1989; chap. 8) for the use of the Stackelberg model in modeling industry entry. Brander and Spencer (1985) use the Cournot-Nash model and the Stackelberg model to model strategic trade practices.

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